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STRESS DISTRIBUTIONS CAUSED BY THREE TYPES OF LOADING ON A CIRCULAR SEMIMONOCOQUE CYLINDER WITH FLEXIBLE RINGS

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STRESS DISTRIBUTIONS CAUSED BY THREE TYPES OF LOADING ON A CIRCULAR SEMIMONOCOQUE CYLINDER WITH FLEXIBLE RINGS

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SUMMARY

Equations are derived for the stress distributions caused by three types of loading on infinitely long, circular, semimonocoque cylinders with flexible rings. These external loads can be said to lie in the sheet-stringer surface of the cylinder; that is, they have no radial components. The results, given as formulas for the stringer loads and shear flows in the shell, are of use in the stress analysis of circular semimonocoque cylinders with cutouts. In order to facilitate this application, the formulas can be used to construct tables of influence coefficients giving stringer loads and shear flows in the neighborhood of each particular external load due to a unit magnitude of that load.

INTRODUCTION

An airplane of semimonocoque construction usually has various openings for doors, cargo hatches, landing-gear mechanisms, and other purposes. The structural design of the airplane near such openings requires knowledge of the stress distribution about cutouts in semimonocoque shells. Stress analysis of semimonocoque structures with cutouts is beyond the scope of engineering beam theory because stress concentrations occur in the neighborhood of the cutouts. It is for this reason that more powerful methods of analysis are sought.

A method for the stress analysis of sheet-stringer panels with cutouts has been presented in reference 1. The method is based on the idea introduced by Cicala (ref. 2) that the effect of the cutout can be reproduced by superposing certain perturbation stress states on the stresses which would occur in the structure without a cutout. Three types of "unit perturbation solutions" were obtained, and it was shown how these solutions could be used to handle cutouts in sheet-stringer panels under axial loads.

It is evident that this approach can be extended to the analysis of stresses around cutouts in circular semimonocoque cylinders. In order to make the extension, however, perturbation solutions for this type of structure must be developed. Accordingly, the purpose of this report is to derive the perturbation solutions for an infinitely long circular cylinder of semimonocoque construction. These solutions are the stress distributions due to three types of loads analogous to those discussed in reference 1. The particular applied perturbation loads for this structure are shown in figure 1. In the analysis for the stresses due to these loads, bending flexibility of the rings in their own planes is taken into consideration.

The use of these perturbation solutions in the stress analysis of circular semimonocoque cylinders with cutouts is described in detail in reference 3.

SYMBOLS

A cross-sectional area of stringers

$A_n = 3B\delta^2 - 1 + \cos n\delta$

$a_{ni} = \dfrac{\Delta_{ii} f_n(i)}{2L \sin \frac{n\delta}{2}} \qquad (n \geq 2)$

$B = \dfrac{E}{G} \dfrac{A}{bt} \dfrac{R^2}{L^2}$

$B_n = 3B\delta^2 + 2(1 - \cos n\delta)$

b arc distance between stringers, $R\delta$

$b_{ni} = -\dfrac{\Delta_{ii} f_n(i)}{2L \sin \frac{n\delta}{2}} \qquad (n \geq 2)$

$C = \dfrac{AR^6}{IL^3 b}$

$$D_n = \frac{2(\beta_n - 1)}{\gamma_n^2}$$

$$D_{rn} = \frac{1}{(rm + n)^2\left[(rm + n)^2 - 1\right]}$$

d_n coefficient in trigonometric series for δ_{0j}

E Young's modulus of elasticity

F_{ij} tangential force on ring i uniformly distributed between stringer j and stringer j + 1

$f_n(i)$ coefficient in trigonometric series for stringer loads

G shear modulus of elasticity

$$H_1(n,\phi) = \sum_{r=-\infty}^{\infty} D_{rn} \cos(rm + n)\phi$$

$$H_2(n,\phi) = \sum_{r=-\infty}^{\infty} (-1)^r D_{rn} \sin(rm + n)\phi$$

I moment of inertia of ring cross section

i longitudinal index, indicating stations (where rings are located) and bays

J_r has the value 1 when r is an integer and has the value 0 when r is not an integer

j circumferential index, indicating stringers and panel rows

k, l, r, s integers

L distance between rings

$M(i,\phi)$ bending moment in ring i

m total number of stringers in the shell, $m \geq 3$

n index of the terms in a trigonometric series

P external concentrated force in the longitudinal direction applied to a stringer at its intersection with a ring

p_{ij} stringer load in stringer j at ring station i

Q external shearing force per unit length applied about a shear panel

q_{ij} shear flow in shear panel (i,j)

R radius to middle surface of sheet

S external force in the longitudinal direction uniformly distributed along that portion of a stringer which lies between adjacent rings

$$S_n = \sum_{r=-\infty}^{\infty} D_{rn}^2$$

$T(i,\phi)$ thrust in ring i

t thickness of sheet

U total stress energy

$V(i,\phi)$ transverse shear in ring i

$\alpha_{1n}, \alpha_{2n}, \alpha_{3n}, \alpha_{4n}$ arbitrary constants

$$\beta_n = 3 + \frac{4 + 3\dfrac{B\delta^2}{\sin^2 \frac{n\delta}{2}}}{12CS_n}$$

$$\gamma_n = -2 + \frac{1 - \frac{3}{2}\dfrac{B\delta^2}{\sin^2 \frac{n\delta}{2}}}{12CS_n}$$

Δ_{ii} second central difference in the i-direction or longitudinal direction, that is, $\Delta_{ii}g(i) = g(i + 1) - 2g(i) + g(i - 1)$

δ central angle between stringers, $2\pi/m$

δ_{rs} Kronecker delta, takes the value 1 when $r = s$ and takes the value 0 when $r \neq s$

$\zeta_n = \pm e^{-\psi_n}$

$\Lambda_{1n}, \Lambda_{2n}$ quantities defined immediately following equation (21)

ϕ angular coordinate for rings

$$\chi_n = \frac{1}{2} \cos^{-1}\left[\frac{\beta_n - 1}{2} - \sqrt{\left(\frac{\beta_n + 1}{2}\right)^2 - \gamma_n^2}\right] \qquad (D_n > 1)$$

$$= \frac{1}{2} \cosh^{-1}\left[\frac{\beta_n - 1}{2} - \sqrt{\left(\frac{\beta_n + 1}{2}\right)^2 - \gamma_n^2}\right] \qquad (D_n < 1)$$

$$\psi_n = \frac{1}{2} \cosh^{-1}\left[\frac{\beta_n - 1}{2} + \sqrt{\left(\frac{\beta_n + 1}{2}\right)^2 - \gamma_n^2}\right]$$

BASIC ASSUMPTIONS

A typical portion of a structure of the type discussed in this report is illustrated in figure 2. It consists of a thin-walled circular cylinder stiffened by stringers in the longitudinal direction and by rings in the circumferential direction. The structure is infinite in length.

The analysis in this report is based on the following assumptions regarding the properties of the structure:

(a) The stringers are uniform and equally spaced around the shell, and the sheet is of constant thickness.

(b) The stringers carry only direct stress. The sheet takes only shear stress which is constant within each bay; thus, the stringer stresses vary linearly between rings.

(c) The rings have a finite bending stiffness in their own planes, but they do not restrain longitudinal displacements of the stringers. The ring bending is inextensional.

(d) The difference between the radius of the neutral axis of the rings and the radius of the middle surface of the sheet is negligible.

ANALYTICAL APPROACH

Equations for the stress distributions arising from the three perturbation loads illustrated in figure 1 are derived in this report. The perturbation load shown in figure 1(a) is a concentrated force in the longitudinal direction applied to one stringer of the shell at a ring station. In figure 1(b) the perturbation load shown is a force in the longitudinal direction uniformly distributed along the portion of one stringer of the shell between adjacent rings. Finally, the perturbation load shown in figure 1(c) consists of forces uniformly distributed on the portions of stringers and rings which border one shear panel of the shell, the forces acting in such a manner as to cause pure shear in that panel. These three loads will be called, respectively, the concentrated, the distributed, and the shear perturbation load.

A detailed explanation of the use of these perturbation loads in the stress analysis of circular semimonocoque cylinders with cutouts is presented in reference 3. Very briefly, the idea is as follows: The cylinder is analyzed by any applicable method with the assumption that the cutout does not exist. Then a system of perturbation loads is applied to the shell without the cutout, and these loads are adjusted in magnitude to force the boundary conditions at the cutout to be satisfied. Stresses are obtained by superposing the perturbation stress distributions on the stresses which would exist in the structure without a cutout. The method is sufficiently versatile to handle the effects of reinforcement about the cutout.

The perturbation solutions will be obtained by use of the principle of minimum complementary energy. This principle states that, among all possible stress distributions in the structure which satisfy equilibrium and the boundary conditions on stress, the distribution that most nearly satisfies compatibility is the one which minimizes the complementary energy π^* where

$$\pi^* = \text{Internal energy} - \text{Work done by surface stresses acting through the prescribed displacements} \qquad (1)$$

Since displacements are not prescribed anywhere on the structure, the second term on the right-hand side of equation (1) is omitted. The complementary energy becomes the internal energy or stress energy of the structure.

In writing the equation for the stress energy, the following factors are considered: the energy of axial distortion of the stringers, the shear energy in the sheet, and the bending energy of the rings in their own planes. In figure 2 are shown the index systems used to designate stringers, rings, bays, and panel rows. Note that the intersection of ring i and stringer j occurs at the lower left-hand corner of shear panel (i,j). Each of the perturbation loads is shown in its positive sense in figure 1. Stringer loads are taken as positive in tension. Shear flows are positive as shown in figure 3. Ring bending moments, shear, and thrusts are placed on the ring element in figure 3 in the positive sense. With this index notation, the stress energy in the structure can be expressed as

$$U = \sum_{i=-\infty}^{\infty} \sum_{j=0}^{m-1} \left[\frac{L}{6AE} \left(p_{ij}^2 + p_{ij} p_{i+1,j} + p_{i+1,j}^2 \right) + \frac{R\delta L}{2Gt} q_{ij}^2 \right] + \sum_{i=-\infty}^{\infty} \int_0^{2\pi} \frac{R}{2EI} M^2(i,\phi) d\phi \qquad (2)$$

where the integration over the length of a stringer between adjacent rings has been carried out.

In the analysis to follow, stringer loads are expressed in the form of a finite trigonometric series. Then, by using the equations of statics, the shear flows and ring bending moments are written in terms of the coefficients of this trigonometric series. The expression for stress energy, equation (2), is minimized with respect to the coefficients of the trigonometric series for stringer loads; then, the expressions for the stringer loads, shear flows, and ring bending moments are substituted into the resulting equation. This process yields a fourth-order finite-difference equation which can be solved for these trigonometric coefficients. The solution is then substituted back into the original expressions for stringer loads, shear flows, and ring moments to yield the desired distributions.

For convenience in application, the significant equations are collected in appendix A.

PERTURBATION LOAD SOLUTIONS

Concentrated Perturbation Load

Expression for stringer loads.- The concentrated perturbation load is shown in figure 1(a); let P represent the magnitude of this load. Since the structure is uniform and infinitely long, half of the load goes into the portion of the structure to the right of the ring station where the load is applied (ring $i = 0$), and half goes to the left of this ring station. Therefore, it can be seen from figure 1(a) that, because of symmetry,

$$p_{ij} = -p_{-i,j} \qquad (i \geqq 1) \quad (3a)$$

$$q_{ij} = q_{-i-1,j} \qquad (i \geqq 0) \quad (3b)$$

$$M(i,\phi) = -M(-i,\phi) \qquad (i \geqq 0) \quad (3c)$$

Consider the right half of the structure, including the ring at $i = 0$. The concentrated perturbation load gives rise to stringer loads which are circumferentially symmetric about stringer $j = 0$ (see fig. 1(a)). Thus the stringer load distribution can be represented by a series of the form

$$p_{ij} = \sum_{n=0}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} f_n(i) \cos n j\delta \qquad (4)$$

where the notation $\sum_{n=0}^{\frac{m}{2} \text{ or } \frac{m-1}{2}}$ means that the summation is carried over n from $n = 0$ to $n = \frac{m}{2}$ if m is even and to $n = \frac{m-1}{2}$ if m is odd.

Evaluation of $f_0(i)$, $f_1(i)$, and $f_n(0)$.- Suppose that equation (4) is multiplied by $\cos l j\delta$ and summed over j from 0 to $m - 1$. This procedure yields

$$\sum_{j=0}^{m-1} p_{ij} \cos l j\delta = \sum_{n=0}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} f_n(i) \sum_{j=0}^{m-1} \cos n j\delta \cos l j\delta$$

The sum over j on the right-hand side is, for $0 \leqq n \leqq \frac{m}{2}$ and $0 \leqq l \leqq \frac{m}{2}$,

$$\sum_{j=0}^{m-1} \cos nj\delta \cos lj\delta = 0 \qquad (l \neq n)$$

$$= \frac{m}{2}\left(1 + \delta_{n0} + \delta_{n,\frac{m}{2}}\right) \qquad (l = n)$$

Thus the coefficients of the trigonometric series in equation (4) are

$$f_n(i) = \frac{2}{m\left(1 + \delta_{n0} + \delta_{n,\frac{m}{2}}\right)} \sum_{j=0}^{m-1} p_{ij} \cos nj\delta \tag{5}$$

It is desirable first of all to evaluate those $f_n(i)$'s which can be found from consideration of the boundary conditions and of the overall equilibrium of the cylinder. Consider the equations of statics for the cylinder as a whole. Satisfaction of equilibrium in the longitudinal direction requires that the sum of the stringer loads at any ring station i must equal one-half of the applied load P. This condition is expressed as

$$\sum_{j=0}^{m-1} p_{ij} = \frac{P}{2}$$

For $n = 0$, equation (5) gives

$$f_0(i) = \frac{1}{m} \sum_{j=0}^{m-1} p_{ij} = \frac{P}{2m} \tag{6}$$

Moment equilibrium gives two equations, one of which is automatically satisfied because of the symmetry of the stringer load distribution around the cylinder. The other moment equation is

$$\sum_{j=0}^{m-1} p_{ij} R \cos j\delta = \frac{PR}{2}$$

For $n = 1$, equation (5) is

$$f_1(i) = \frac{2}{m} \sum_{j=0}^{m-1} p_{ij} \cos j\delta = \frac{P}{m} \tag{7}$$

On substituting the values of $f_0(i)$ and $f_1(i)$ given in equations (6) and (7), respectively, into equation (4), there results

$$p_{ij} = \frac{P}{2m} + \frac{P}{m} \cos j\delta + \sum_{n=2}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} f_n(i) \cos nj\delta \tag{8}$$

Consider now the boundary condition at ring station $i = 0$. The stringer loads here are

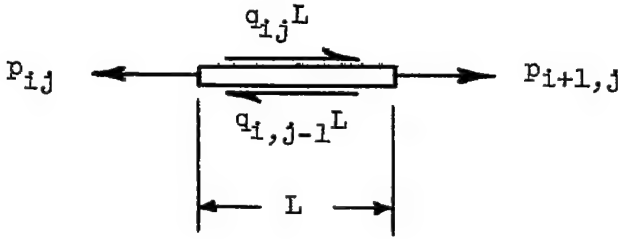
$$p_{0j} = \frac{P}{2} \delta_{0j}$$

and substitution of this expression into equation (5) yields

$$f_n(0) = \frac{P}{m\left(1 + \delta_{n0} + \delta_{n,\frac{m}{2}}\right)} \qquad \left(0 \leq n \leq \frac{m}{2}\right) \tag{9}$$

The equations of equilibrium and the boundary condition at $i = 0$ have been used to obtain certain of the coefficients of the trigonometric series for stringer loads. The remainder of the coefficients are found by use of the principle of minimum complementary energy, and this is the next step in the solution.

Expressions for shear flows and ring bending moments.- In order to use the principle of minimum complementary energy, the shear flows and ring bending moments must be found in terms of the trigonometric coefficients $f_n(i)$. Shear flows are determined by the consideration of the equations of statics of a portion of any stringer j between two adjacent rings i and $i + 1$. The forces on this free body are shown in the sketch below:



Equilibrium of these forces requires that

$$p_{i+1,j} - p_{ij} + \left(q_{ij} - q_{i,j-1}\right)L = 0 \tag{10}$$

Substitution of equation (8) into equation (10) yields

$$q_{ij} - q_{i,j-1} = -\frac{1}{L}\sum_{n=2}^{\frac{m}{2}\text{ or }\frac{m-1}{2}} \left[f_n(i+1) - f_n(i)\right]\cos nj\delta \tag{11}$$

In order to find q_{ij}, replace j with a dummy index k and sum both sides of this equation over k from $k = 1$ to $k = j$; that is, write

$$\sum_{k=1}^{j}\left(q_{ik} - q_{i,k-1}\right) = -\frac{1}{L}\sum_{n=2}^{\frac{m}{2}\text{ or }\frac{m-1}{2}} \left[f_n(i+1) - f_n(i)\right]\sum_{k=1}^{j}\cos nk\delta$$

When the indicated summations over k have been carried out, the following equation is obtained:

$$q_{ij} - q_{i0} = -\frac{1}{L}\sum_{n=2}^{\frac{m}{2}\text{ or }\frac{m-1}{2}} \left[f_n(i+1) - f_n(i)\right]\left[\frac{\sin n\left(j + \frac{1}{2}\right)\delta}{2\sin\frac{n\delta}{2}} - \frac{1}{2}\right]$$

The term q_{i0} can be found from the condition that the total torque on the section is zero. The resulting expression for shear flows is

$$q_{ij} = -\sum_{n=2}^{\frac{m}{2}\text{ or }\frac{m-1}{2}} \frac{f_n(i+1) - f_n(i)}{2L\sin\frac{n\delta}{2}}\sin n\left(j + \frac{1}{2}\right)\delta \tag{12}$$

Bending moments are caused in each ring by a tangential loading which develops because of the difference in shear flow in the sheet on either side of the ring. The tangential load on ring i has the value

$$q_{ij} - q_{i-1,j} = -\sum_{n=2}^{\frac{m}{2}\text{ or }\frac{m-1}{2}} \frac{\Delta_{ii} f_n(i)}{2L\sin\frac{n\delta}{2}}\sin n\left(j + \frac{1}{2}\right)\delta \tag{13}$$

In appendix B, this load is applied to a circular ring and the bending moment in the ring is derived. This procedure results in the following moment in ring i (see eq. (B9)):

$$M(i,\phi) = -\sum_{n=2}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} \frac{R^2 m}{2\pi L} \Delta_{ii} f_n(i) H_1(n,\phi) \tag{14}$$

where

$$H_1(n,\phi) = \sum_{r=-\infty}^{\infty} \frac{\cos(rm + n)\phi}{(rm + n)^2 \left[(rm + n)^2 - 1\right]}$$

The sign convention for the moment is illustrated in figure 3; the convention for measuring angle ϕ is shown in figure 4(a).

Energy analysis.- Now the stringer loads, shear flows, and ring bending moments have been expressed in terms of the coefficients $f_n(i)$. The stringer loads are given in equation (8), the shear flows in equation (12), and the bending moments in equation (14). These equations are used in the minimization of the stress energy of the cylinder with respect to $f_n(i)$.

By virtue of the symmetry properties of this problem expressed in equations (3a), (3b), and (3c), the energy in the structure to the left of ring i = 0 is the same as the energy to the right of ring i = 0. Thus, equation (2) becomes

$$U = 2\sum_{i=0}^{\infty} \sum_{j=0}^{m-1} \left[\frac{L}{6AE}\left(p_{ij}^2 + p_{ij}p_{i+1,j} + p_{i+1,j}^2\right) + \frac{R\delta L}{2Gt} q_{ij}^2\right] +$$

$$2\sum_{i=1}^{\infty} \int_0^{2\pi} \frac{R}{2EI} M^2(i,\phi)\,d\phi$$

Note that $M(0,\phi)$ is identically zero because there is no difference in shear flow across ring i = 0 and, therefore, no tangential load acts on this ring.

Minimization of the stress energy with respect to $f_n(i)$ results in the following equation:

$$\frac{\partial U}{\partial f_n(i)} = 0$$

$$= \sum_{j=0}^{m-1} \left[\frac{L}{6AE}\left(p_{i+1,j} + 4p_{ij} + p_{i-1,j}\right)\frac{\partial p_{ij}}{\partial f_n(i)} + \frac{R\delta L}{Gt}\left(q_{ij}\frac{\partial q_{ij}}{\partial f_n(i)} + q_{i-1,j}\frac{\partial q_{i-1,j}}{\partial f_n(i)}\right)\right] + \int_0^{2\pi} \frac{R}{EI}\left[M(i+1,\phi)\frac{\partial M(i+1,\phi)}{\partial f_n(i)} + M(i,\phi)\frac{\partial M(i,\phi)}{\partial f_n(i)} + M(i-1,\phi)\frac{\partial M(i-1,\phi)}{\partial f_n(i)}\right] d\phi \qquad (15)$$

The coefficients $f_0(i)$ and $f_1(i)$ are known already for all values of i, and $f_n(0)$ is known for $0 \leq n \leq \frac{m}{2}$. Thus equation (15) needs only to be considered for $i \geq 1$ and $n \geq 2$. The expressions for the stringer loads, shear flows, and ring bending moments are substituted into equation (15). Then the following definite sums are needed (these can be obtained by the procedure outlined in ref. 4):

$$\sum_{j=0}^{m-1} \cos nj\delta = 0 \qquad (0 < n < m) \qquad (16)$$

and for the integers n and l restricted to the range $1 \leq n \leq \frac{m}{2}$ and $1 \leq l \leq \frac{m}{2}$,

$$\left.\begin{aligned} \sum_{j=0}^{m-1} \cos lj\delta \cos nj\delta &= 0 && (l \neq n) \\ &= \frac{m}{2}\left(1 + \delta_{n,\frac{m}{2}}\right) && (l = n) \end{aligned}\right\} \qquad (17)$$

and

$$\left.\begin{aligned} \sum_{j=0}^{m-1} \sin l\left(j + \tfrac{1}{2}\right)\delta \sin n\left(j + \tfrac{1}{2}\right)\delta &= 0 && (l \neq n) \\ &= \frac{m}{2}\left(1 + \delta_{n,\frac{m}{2}}\right) && (l = n) \end{aligned}\right\} \qquad (18)$$

The following definite integral, which is derived in appendix C, is also needed:

$$\left.\begin{aligned} \int_0^{2\pi} H_l(l,\phi)H_l(n,\phi)d\phi &= 0 \qquad (l \neq n) \\ &= \pi S_n\left(1 + \delta_{n,\frac{m}{2}}\right) \qquad (l = n) \end{aligned}\right\} \tag{19}$$

where

$$S_n = \sum_{r=-\infty}^{\infty} D_{rn}^2 = \sum_{r=-\infty}^{\infty} \frac{1}{(rm + n)^4\left[(rm + n)^2 - 1\right]^2}$$

and where n and l are restricted to $2 \leqq n \leqq \frac{m}{2}$ and $2 \leqq l \leqq \frac{m}{2}$. A closed form of S_n is presented in appendix C but the series form converges so rapidly that it is usually more convenient than the closed form for use in calculations.

After substitution of the expressions for stringer loads, shear flows, and ring moments into equation (15), the use of these definite sums (16), (17), and (18), and definite integral (19) results in the following equations which express the condition of minimum stress energy:

For $i = 1$,

$$f_n(3) + 2\gamma_n f_n(2) + (2\beta_n - 1)f_n(1) + 2(\gamma_n + 1)f_n(0) = 0 \tag{20a}$$

and, for $i \geqq 2$,

$$f_n(i+2) + 2\gamma_n f_n(i+1) + 2\beta_n f_n(i) + 2\gamma_n f_n(i-1) + f_n(i-2) = 0 \tag{20b}$$

where

$$\gamma_n = -2 + \frac{1 - \frac{3}{2}\dfrac{B\delta^2}{\sin^2 \frac{n\delta}{2}}}{12CS_n}$$

$$\beta_n = 3 + \frac{4 + 3\dfrac{B\delta^2}{\sin^2 \dfrac{n\delta}{2}}}{12CS_n}$$

$$B = \frac{E}{G}\frac{A}{bt}\frac{R^2}{L^2}$$

$$C = \frac{AR^6}{IL^3b}$$

Solution of finite-difference equation.- Equation (20b) is a fourth-order finite-difference equation with constant coefficients. (Note that the symbol i represents the index of the rings and bays and should not be confused with the usual notation for $\sqrt{-1}$ which never appears in this report.) Equation (20b) corresponds exactly with equation (24) of reference 5. The general solution is presented on pages 23 to 26 of reference 5 and on pages 28 and 29 of reference 6. It may be written as

$$f_n(i) = \left(\pm e^{-\psi_n}\right)^i \left[\alpha_{1n}\Lambda_{1n}(i) + \alpha_{2n}\Lambda_{2n}(i)\right] + \left(\pm e^{\psi_n}\right)^i \left[\alpha_{3n}\Lambda_{1n}(i) + \alpha_{4n}\Lambda_{2n}(i)\right] \qquad (n \geqq 2) \qquad (21)$$

where the upper sign is used when $\gamma_n < 0$ and the lower sign when $\gamma_n > 0$. The Λ's are as follows:

For $D_n = \dfrac{2(\beta_n - 1)}{\gamma_n^2} > 1$,

$$\Lambda_{1n}(i) = \cos i\chi_n$$

$$\Lambda_{2n}(i) = \sin i\chi_n$$

where

$$\chi_n = \frac{1}{2}\cos^{-1}\left[\frac{\beta_n - 1}{2} - \sqrt{\left(\frac{\beta_n + 1}{2}\right)^2 - \gamma_n^2}\right]$$

for $D_n < 1$,

$$\Lambda_{1n}(i) = \cosh i\chi_n$$

$$\Lambda_{2n}(i) = \sinh i\chi_n$$

where

$$\chi_n = \frac{1}{2}\cosh^{-1}\left[\frac{\beta_n - 1}{2} - \sqrt{\left(\frac{\beta_n + 1}{2}\right)^2 - \gamma_n^2}\right]$$

and, for $D_n = 1$,

$$\Lambda_{1n} = 1$$

$$\Lambda_{2n} = i$$

In the inverse trigonometric and hyperbolic functions, the principal values are used. The argument ψ_n of the exponential function is given by the positive branch of

$$\psi_n = \frac{1}{2}\cosh^{-1}\left[\frac{\beta_n - 1}{2} + \sqrt{\left(\frac{\beta_n + 1}{2}\right)^2 - \gamma_n^2}\right]$$

At a large longitudinal distance from the applied load, the stringer loads should approach the elementary distribution given by the first two terms of equation (8); consequently, for $n \geq 2$, $f_n(i) \to 0$ as $i \to \infty$. The first term on the right-hand side of equation (21) satisfies this condition; however, the second term does not and, hence, must be omitted. The solutions, then, that are compatible with the boundary conditions at infinity are:

$$f_n(i) = \zeta_n{}^i\left[\alpha_{1n}\Lambda_{1n}(i) + \alpha_{2n}\Lambda_{2n}(i)\right] \qquad (n \geq 2) \quad (22)$$

where

$$\zeta_n = \pm e^{-\psi_n}$$

Now the arbitrary constants α_{1n} and α_{2n} are determined. The first α_{1n} is obtained immediately. Substitution of $i = 0$ into equation (22) and use of equation (9) to evaluate $f_n(0)$ yields

$$f_n(0) = \alpha_{1n} = \frac{P}{m\left(1 + \delta_{n,\frac{m}{2}}\right)} \qquad (n \geq 2) \qquad (23)$$

Substitution of equations (23) and (22) into the boundary equation (20a) yields

$$\alpha_{2n} = -\frac{\Theta_{1n} + 2(\gamma_n + 1)}{\Theta_{2n}} \frac{P}{m\left(1 + \delta_{n,\frac{m}{2}}\right)}$$

where

$$\Theta_{sn} = \zeta_n^3 \Lambda_{sn}(3) + 2\gamma_n \zeta_n^2 \Lambda_{sn}(2) + (2\beta_n - 1)\zeta_n \Lambda_{sn}(1) \qquad (s = 1,2)$$

The solution for the concentrated perturbation load is now complete since the coefficients $f_n(i)$ are completely defined and may be substituted into equation (8) to give the stringer loads. The shear flows can be found from equation (12); however, once the stringer loads are known, it is simpler to calculate the shear flows by the use of the equations of statics. Because of symmetry, the shear flows in shear panels adjacent to stringer $j = 0$ are given by

$$q_{i0} = -q_{i,-1} = \frac{p_{i0} - p_{i+1,0}}{2L}$$

All the other shear flows can be found by the use of equation (10). If desired, the moment distribution in the rings can be computed from equation (14) and the thrust and transverse shear in the rings can be found from the formulas given in appendix B.

Distributed Perturbation Load

Expression for stringer loads.- The distributed perturbation load is shown in figure 1(b); let S represent the magnitude of the total force distributed along stringer $j = 0$ between rings $i = 0$ and $i = 1$. From figure 1(b) it is seen that

$$p_{ij} = -p_{-i+1,j} \qquad (i \geqq 1) \qquad (24a)$$

$$q_{ij} = q_{-i,j} \qquad (i \geqq 1) \qquad (24b)$$

$$M(i,\phi) = -M(-i+1,\phi) \qquad (i \geqq 1) \qquad (24c)$$

At ring station $i = 1$ and to the right of this ring, the stringer loads can be represented by a trigonometric series of exactly the same form as equation (4)

$$p_{ij} = \sum_{n=0}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} f_n(i)\cos nj\delta \qquad (25)$$

except now $i \geqq 1$, and the coefficients $f_n(i)$ are different from those obtained for the preceding case of the concentrated load.

Evaluation of $f_0(i)$ and $f_1(i)$.- As in the preceding case, the first two coefficients $f_0(i)$ and $f_1(i)$ can be obtained from the equations of statics, and the results are the same as before. Equation (25) becomes

$$p_{ij} = \frac{S}{2m} + \frac{S}{m}\cos j\delta + \sum_{n=2}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} f_n(i)\cos nj\delta \qquad (i \geqq 1) \qquad (26)$$

With the concentrated perturbation load, all the coefficients $f_n(0)$ were easily found because the stringer load distribution at ring station $i = 0$ was known. Here no such distribution is known. In order to determine the boundary condition at bay $i = 0$, the effect of the distributed perturbation load on the equilibrium of portions of stringers in this bay must be investigated.

Expressions for shear flows and ring bending moments.- Away from bay $i = 0$ the shear flows and ring bending moments are of the same form

as for the concentrated load. The following expression for the shear flows is obtained by use of equation (10):

$$q_{ij} = -\sum_{n=2}^{\frac{m}{2}\text{ or }\frac{m-1}{2}} \frac{f_n(i+1) - f_n(i)}{2L \sin\frac{n\delta}{2}} \sin n\left(j + \frac{1}{2}\right)\delta \quad (i \geqq 1) \quad (27)$$

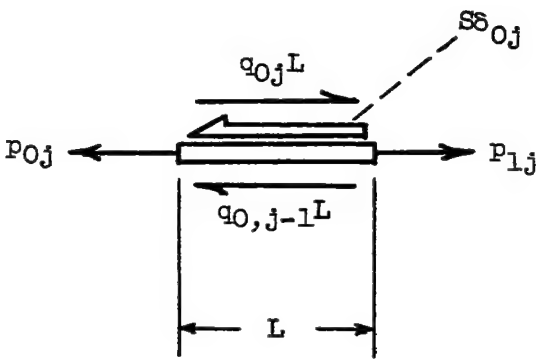
The ring bending moments are obtained in appendix B as

$$M(i,\phi) = -\sum_{n=2}^{\frac{m}{2}\text{ or }\frac{m-1}{2}} \frac{R^2 m}{2\pi L} \Delta_{ii} f_n(i) H_1(n,\phi) \qquad (i \geqq 2) \quad (28)$$

where

$$H_1(n,\phi) = \sum_{r=-\infty}^{\infty} \frac{\cos(rm + n)\phi}{(rm + n)^2\left[(rm + n)^2 - 1\right]}$$

The applied force in bay $i = 0$ may be written as $S\delta_{0j}$. Consider, now, the equilibrium of a portion of any stringer j between ring $i = 0$ and ring $i = 1$. The forces on this free body are shown in the sketch below:



Equilibrium of these forces requires that

$$P_{1j} - P_{0j} + \left(q_{0j} - q_{0,j-1}\right)L - S\delta_{0j} = 0$$

Because of the antisymmetry property expressed in equation (24a), the equilibrium equation becomes

$$2p_{1j} + \left(q_{0j} - q_{0,j-1}\right)L - S\delta_{0j} = 0 \tag{29}$$

It is convenient, now, to expand the Kronecker delta δ_{0j} in a finite trigonometric series,

$$\delta_{0j} = \sum_{n=0}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} d_n \cos nj\delta \tag{30}$$

Multiplying through by $\cos lj\delta$ and summing over j from 0 to $m - 1$, yields the trigonometric coefficients d_n. The result is

$$d_n = \frac{2}{m\left(1 + \delta_{n0} + \delta_{n,\frac{m}{2}}\right)} \tag{31}$$

Substitution of the expression for stringer loads, equation (26), and the trigonometric expansion for δ_{0j}, equation (30), into the equilibrium equation (29) yields

$$q_{0j} - q_{0,j-1} = \frac{1}{L} \sum_{n=2}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} \left[Sd_n - 2f_n(1)\right] \cos nj\delta$$

In order to find q_{0j}, this equation can be treated in the same manner as equation (11); that is, replace j by a dummy index k, sum from $k = 1$ to $k = j$, and then use the condition that the total torque on a cross section in bay $i = 0$ must be zero. This procedure results in the following expression for the shear flows in bay $i = 0$:

$$q_{0j} = \sum_{n=2}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} \frac{\frac{1}{2} Sd_n - f_n(1)}{L \sin \frac{n\delta}{2}} \sin n\left(j + \frac{1}{2}\right)\delta \tag{32}$$

The expression for the bending moment in rings $i = 1$ and $i = 0$ is yet to be found, as this expression differs from that for the moment in the rest of the rings given in equation (28). The moment in ring $i = 0$ is the same in magnitude as that in ring $i = 1$ but opposite in sign.

The tangential loading on ring i = 1 is given by

$$q_{1j} - q_{0j} = - \sum_{n=2}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} \frac{f_n(2) - 3f_n(1) + Sd_n}{2L \sin \frac{n\delta}{2}} \sin n\left(j + \frac{1}{2}\right)\delta$$

By analogy with equations (13) and (14), then, the bending moment in ring i = 1 can be written as

$$M(1,\phi) = - \sum_{n=2}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} \frac{R^2 m}{2\pi L} \left[f_n(2) - 3f_n(1) + Sd_n\right] H_1(n,\phi) \qquad (33)$$

All the stringer loads, shear flows, and ring bending moments have now been expressed in terms of the coefficients $f_n(i)$. The stringer loads are given in equation (26), the shear flows in equations (27) and (32), and the ring moments in equations (28) and (33). It remains to substitute these expressions into the equation obtained from minimization of the stress energy of the cylinder with respect to $f_n(i)$.

Energy analysis.- By virtue of the symmetry properties in this problem, given in equations (24), the energy in the structure to the right of bay i = 0 equals the energy to the left of this bay. Equation (2) for the stress energy can be written

$$U = \sum_{j=0}^{m-1}\left(\frac{L}{6AE} p_{1j}^2 + \frac{R\delta L}{2Gt} q_{0j}^2\right) + 2 \sum_{i=1}^{\infty} \sum_{j=0}^{m-1} \left[\frac{L}{6AE}\left(p_{ij}^2 + p_{ij}p_{i+1,j} + p_{i+1,j}^2\right) + \right.$$

$$\left. \frac{R\delta L}{2Gt} q_{ij}^2\right] + 2 \sum_{i=1}^{\infty} \int_0^{2\pi} \frac{R}{2EI} M^2(i,\phi)d\phi \qquad (34)$$

Minimization of the stress energy with respect to $f_n(i)$ results in the following equations:

$$\frac{\partial U}{\partial f_n(1)} = 0$$

$$= \sum_{j=0}^{m-1}\left[\frac{L}{6AE}\left(3p_{1j} + p_{2j}\right)\frac{\partial p_{1j}}{\partial f_n(1)} + \frac{R\delta L}{2Gt}\left(2q_{1j}\frac{\partial q_{1j}}{\partial f_n(1)} + q_{0j}\frac{\partial q_{0j}}{\partial f_n(1)}\right)\right] +$$

$$\int_0^{2\pi}\frac{R}{EI}\left[M(2,\phi)\frac{\partial M(2,\phi)}{\partial f_n(1)} + M(1,\phi)\frac{\partial M(1,\phi)}{\partial f_n(1)}\right]d\phi \qquad (35)$$

and

$$\frac{\partial U}{\partial f_n(i)} = 0$$

$$= \sum_{j=0}^{m-1}\left[\frac{L}{6AE}\left(p_{i+1,j} + 4p_{ij} + p_{i-1,j}\right)\frac{\partial p_{ij}}{\partial f_n(i)} + \frac{R\delta L}{Gt}\left(q_{ij}\frac{\partial q_{ij}}{\partial f_n(i)} + q_{i-1,j}\frac{\partial q_{i-1,j}}{\partial f_n(i)}\right)\right] + \int_0^{2\pi}\frac{R}{EI}\left[M(i+1,\phi)\frac{\partial M(i+1,\phi)}{\partial f_n(i)} + M(i,\phi)\frac{\partial M(i,\phi)}{\partial f_n(i)} + M(i-1,\phi)\frac{\partial M(i-1,\phi)}{\partial f_n(i)}\right]d\phi \qquad (i \geq 2) \qquad (36)$$

Note that equation (36) is the same as equation (15), except here it is valid only for $i \geq 2$.

The stringer loads, shear flows, and ring moments are substituted into equations (35) and (36), and then the definite sums and definite integral derived in the preceding section are used to simplify these equations. After simplification, the following equations result:

For $i = 1$,

$$f_n(3) + (2\gamma_n - 1)f_n(2) + 2(\beta_n - \gamma_n)f_n(1) = 8d_n\left(\frac{\beta_n - 4\gamma_n - 2}{3}\right) \qquad (37a)$$

for $i = 2$,

$$f_n(4) + 2\gamma_n f_n(3) + 2\beta_n f_n(2) + (2\gamma_n - 1)f_n(1) = -Sd_n \qquad (37b)$$

and, for $i \geq 3$,

$$f_n(i+2) + 2\gamma_n f_n(i+1) + 2\beta_n f_n(i) + 2\gamma_n f_n(i-1) + f_n(i-2) = 0 \qquad (37c)$$

Solution of finite-difference equation.- Equation (37c) is the same as equation (20b); therefore, the solution to equation (37c) is

$$f_n(i) = \zeta_n{}^i\left[\alpha_{1n}\Lambda_{1n}(i) + \alpha_{2n}\Lambda_{2n}(i)\right] \qquad (n \geq 2) \qquad (38)$$

which is the same as equation (22) except for the values of the arbitrary constants α_{1n} and α_{2n}. These constants are found by the substitution of the solution (38) into equations (37a) and (37b). This procedure yields two simultaneous algebraic equations in α_{1n} and α_{2n}, and their solution gives

$$\alpha_{1n} = \frac{\Gamma_{2n}\dfrac{\beta_n - 4\gamma_n - 2}{3} + \Omega_{2n}}{\Omega_{1n}\Gamma_{2n} - \Gamma_{1n}\Omega_{2n}} \frac{2S}{m\left(1 + \delta_{n,\frac{m}{2}}\right)}$$

$$\alpha_{2n} = -\frac{\Omega_{1n} + \Gamma_{1n}\dfrac{\beta_n - 4\gamma_n - 2}{3}}{\Omega_{1n}\Gamma_{2n} - \Gamma_{1n}\Omega_{2n}} \frac{2S}{m\left(1 + \delta_{n,\frac{m}{2}}\right)}$$

where d_n, the coefficient in the trigonometric series for the Kronecker delta δ_{0j}, has been replaced by its value as given in equation (31), and where the Ω's and Γ's are given by

$$\Omega_{sn} = \zeta_n{}^3\Lambda_{sn}(3) + (2\gamma_n - 1)\zeta_n{}^2\Lambda_{sn}(2) + 2(\beta_n - \gamma_n)\zeta_n\Lambda_{sn}(1) \qquad (s = 1,2) \qquad (39a)$$

$$\Gamma_{sn} = \zeta_n{}^4\Lambda_{sn}(4) + 2\gamma_n\zeta_n{}^3\Lambda_{sn}(3) + 2\beta_n\zeta_n{}^2\Lambda_{sn}(2) + (2\gamma_n - 1)\zeta_n\Lambda_{sn}(1) \qquad (s = 1,2) \qquad (39b)$$

The coefficients $f_n(i)$ are now defined for the distributed perturbation load and may be substituted into equation (26) to give the stringer loads. The shear flows can be found from equations (27) and (32), but, again, once the stringer loads are known shear flows can easily be found by use of the equations of statics. The shear flow in the panels adjacent to stringer $j = 0$ can be found by considering symmetry:

In bay $i = 0$

$$q_{00} = -q_{0,-1} = \frac{S - 2p_{10}}{2L}$$

and, outside of bay $i = 0$,

$$q_{i0} = -q_{i,-1} = \frac{p_{i0} - p_{i+1,0}}{2L} \qquad (i \geqq 1)$$

The other shear flows are found from equation (10), as before. If desired, the ring moments can be obtained from equations (28) and (33) and the ring thrust and transverse shear can be found from the equations given in appendix B.

Shear Perturbation Load

Expression for stringer loads.- The shear perturbation load is shown in figure 1(c). The magnitude of the load per unit length applied along the stringers and rings bordering shear panel (0,0) will be represented by Q. From figure 1(c) it is seen that the longitudinal symmetry properties in this case are the same as those for the case of the distributed perturbation load given by equations (24).

The shear perturbation load is self-equilibrating and gives rise to stringer loads which are antisymmetric about panel row $j = 0$. For $i \geqq 1$, the stringer loads may be represented by

$$p_{ij} = \sum_{n=2}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} f_n(i) \sin n\left(j - \frac{1}{2}\right)\delta \qquad (40)$$

where the coefficients $f_n(i)$ are different from those in the two preceding cases. The term corresponding to $n = 1$ vanishes because it represents an elementary bending stringer-load distribution, and the shear perturbation load does not require this distribution for overall equilibrium.

Expressions for shear flows and ring bending moments.- None of the coefficients $f_n(i)$ in the trigonometric series (40) can be found from the equations of statics. Furthermore, the boundary condition at bay $i = 0$ must be determined from a consideration of the effect that the shear perturbation load has on the equilibrium of the portions of stringers in bay $i = 0$ and on the bending moment in the rings bounding this bay. Thus the energy approach must be used immediately and the first step in this approach is to write the shear flows and ring moments in terms of $f_n(i)$, the coefficients of the trigonometric series for the stringer loads, equation (40).

Outside of bay $i = 0$, the satisfaction of the equations of statics for the portions of stringers between adjacent rings yields equation (10), the same as in the two preceding cases. Substituting equation (40) for the stringer loads into the equilibrium equation (10) and following the same procedure used to obtain equation (12) yields the expression for the shear flows due to the shear perturbation load:

$$q_{ij} = \sum_{n=2}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} \frac{f_n(i+1) - f_n(i)}{2L \sin \frac{n\delta}{2}} \cos nj\delta \qquad (i \geq 1) \quad (41)$$

The tangential loadings on the rings to the right of ring $i = 1$ are

$$q_{ij} - q_{i-1,j} = \sum_{n=2}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} \frac{\Delta_{11} f_n(i)}{2L \sin \frac{n\delta}{2}} \cos nj\delta$$

In appendix B this load is applied to a circular ring and the following expression for the moment in the ring is obtained (see eq. (B13)):

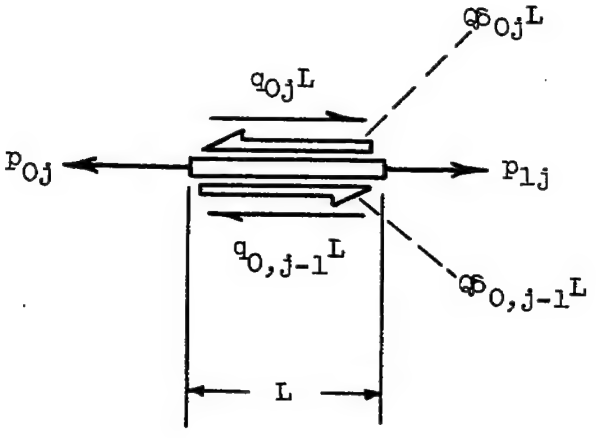
$$M(i,\phi) = - \sum_{n=2}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} \frac{R^2 m}{2\pi L} \Delta_{11} f_n(i) H_2(n,\phi) \qquad (i \geq 2) \quad (42)$$

where

$$H_2(n,\phi) = \sum_{r=-\infty}^{\infty} (-1)^r \frac{\sin(rm + n)\phi}{(rm + n)^2 \left[(rm + n)^2 - 1\right]}$$

The convention for measuring angle ϕ here is a little different than before and is illustrated in figure 4(b).

Now, the shear flows in bay i = 0 and the bending moments in the rings bordering bay i = 0 must be found. Consider the shear flows in this central bay. The shear perturbation loading applied at bay i = 0 may be written $Q\delta_{0j}$. Then the forces on the portion of any stringer j between ring i = 0 and ring i = 1 are as shown in the sketch below:



Equilibrium of these forces requires that

$$p_{1j} - p_{0j} + \left(q_{0j} - q_{0,j-1}\right)L + Q\left(\delta_{0,j-1} - \delta_{0j}\right)L = 0$$

Because of the antisymmetry property, equation (24a), the equation of equilibrium becomes

$$2p_{1j} + \left(q_{0j} - q_{0,j-1}\right)L + Q\left(\delta_{0,j-1} - \delta_{0j}\right)L = 0 \qquad (43)$$

The substitution of the stringer loads, equation (40), into the equilibrium equation (43), and the introduction of the trigonometric expansion for the Kronecker delta δ_{0j}, equation (30), yields the following equation:

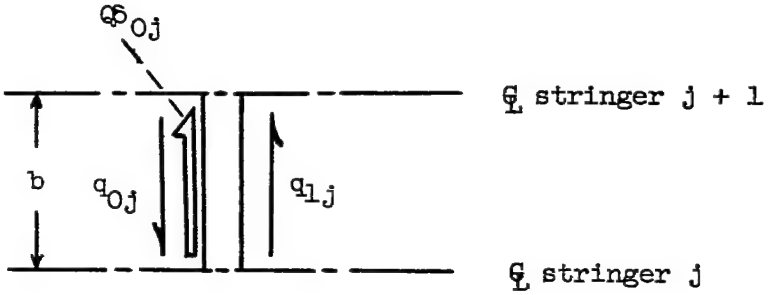
$$q_{0j} - q_{0,j-1} = -\frac{1}{L}\sum_{n=2}^{\frac{m}{2}\text{ or }\frac{m-1}{2}} 2f_n(1)\sin n\left(j - \frac{1}{2}\right)\delta - Qd_1\left[\cos(j-1)\delta - \cos j\delta\right] -$$

$$\sum_{n=2}^{\frac{m}{2}\text{ or }\frac{m-1}{2}} Qd_n\left[\cos n(j-1)\delta - \cos nj\delta\right]$$

Now q_{0j} can be found by replacing j with a dummy index k, summing over k from $k = 1$ to $k = j$, and using the condition that the torque on a cross section within bay $i = 0$ balances the applied torque. This procedure results in the following equation for the shear flow in the central bay:

$$q_{0j} = Qd_0 + Qd_1 \cos j\delta + \sum_{n=2}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} \left(\frac{f_n(1)}{L \sin \frac{n\delta}{2}} + Qd_n \right) \cos nj\delta \qquad (44)$$

Consider the bending moment in rings $i = 1$ and $i = 0$. The moment in ring $i = 0$ is identical in magnitude to the moment in ring $i = 1$ but of opposite sign. The tangential loading per unit arc length on the portion of ring $i = 1$ between stringer j and stringer $j + 1$ is illustrated in the sketch:



Adding these tangential loads and introducing the series expansions for q_{0j} and q_{1j} and for δ_{0j} yields the total load per unit arc length on ring $i = 1$:

$$q_{1j} - q_{0j} + q_{\delta 0j} = \sum_{n=2}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} \frac{f_n(2) - 3f_n(1)}{2L \sin \frac{n\delta}{2}} \cos nj\delta$$

By analogy with equations (13) and (14) the bending moment in ring $i = 1$ is

$$M(1,\phi) = - \sum_{n=2}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} \frac{R^2 m}{2\pi L} \left[f_n(2) - 3f_n(1) \right] H_2(n,\phi) \qquad (45)$$

Expressions for stringer loads, shear flows, and ring moments have been written in terms of the coefficients $f_n(i)$. The stringer loads are given in equation (40), the shear flows in equations (41) and (44), and ring moments in equations (42) and (45). These expressions are ready to be substituted into the equation which results from minimizing the stress energy with respect to $f_{\bar{n}}(i)$.

Energy analysis.- Because the longitudinal symmetry relations which exist for the distributed perturbation load, equations (24), also exist in the case of the shear perturbation load, the stress-energy expression used in the distributed-load problem can be used here. The expressions obtained on minimizing this stress energy, equations (35) and (36), are also applicable here. So the stringer loads, shear flows, and ring moments just derived are substituted into equations (35) and (36). At this stage in the two preceding cases, certain definite sums and a definite integral were introduced to simplify the equations. A similar procedure is followed here.

The definite sums which are of interest are

$$\sum_{j=0}^{m-1} \sin n\left(j - \frac{1}{2}\right)\delta = 0$$

and for the integers n and l restricted to the range $1 \leqq n \leqq \frac{m}{2}$ and $1 \leqq l \leqq \frac{m}{2}$

$$\sum_{j=0}^{m-1} \cos lj\delta \cos nj\delta = 0 \qquad (l \neq n)$$

$$= \frac{m}{2}\left(1 + \delta_{n,\frac{m}{2}}\right) \qquad (l = n)$$

and

$$\sum_{j=0}^{m-1} \sin l\left(j - \frac{1}{2}\right)\delta \sin n\left(j - \frac{1}{2}\right)\delta = 0 \qquad (l \neq n)$$

$$= \frac{m}{2}\left(1 + \delta_{n,\frac{m}{2}}\right) \qquad (l = n)$$

The definite integral, which is derived in appendix C, is

$$\int_0^{2\pi} H_2(l,\phi)H_2(n,\phi)d\phi = 0 \qquad (l \neq n)$$

$$= \pi S_n\left(1 + \delta_{n,\frac{m}{2}}\right) \qquad (l = n)$$

where n and l are restricted to $2 \leqq n \leqq \frac{m}{2}$ and $2 \leqq l \leqq \frac{m}{2}$. After simplification the following equations result:

For $i = 1$,

$$f_n(3) + (2\gamma_n - 1)f_n(2) + 2(\beta_n - \gamma_n)f_n(1) = -2LQd_n\left(\frac{\beta_n - 4\gamma_n - 11}{3}\right)\sin\frac{n\delta}{2} \tag{46a}$$

for $i = 2$,

$$f_n(4) + 2\gamma_n f_n(3) + 2\beta_n f_n(2) + (2\gamma_n - 1)f_n(1) = 0 \tag{46b}$$

and, for $i \geqq 3$,

$$f_n(i+2) + 2\gamma_n f_n(i+1) + 2\beta_n f_n(i) + 2\gamma_n f_n(i-1) + f_n(i-2) = 0 \tag{46c}$$

Solution of finite-difference equation.- Equation (46c) is the same finite-difference equation for which the solution is written in the two preceding sections. Substitution of this solution, equation (38), into equations (46a) and (46b) gives two simultaneous algebraic equations for α_{1n} and α_{2n}, the arbitrary constants. Solution of this system yields

$$\alpha_{1n} = -\frac{\Gamma_{2n}\frac{\beta_n - 4\gamma_n - 11}{3}\sin\frac{n\delta}{2}}{\Omega_{1n}\Gamma_{2n} - \Gamma_{1n}\Omega_{2n}}\,\frac{4QL}{m\left(1 + \delta_{n,\frac{m}{2}}\right)}$$

$$\alpha_{2n} = \frac{\Gamma_{1n} \frac{\beta_n - 4\gamma_n - 11}{3} \sin \frac{n\delta}{2}}{\Omega_{1n}\Gamma_{2n} - \Gamma_{1n}\Omega_{2n}} \frac{4QL}{m\left(1 + \delta_{n,\frac{m}{2}}\right)}$$

The Ω's and Γ's in this case are precisely the same as in the preceding case of the distributed perturbation load; Ω_{sn} is given by equation (39a) and Γ_{sn} by equation (39b).

With the coefficients $f_n(i)$ known for the shear perturbation load, the stringer loads are obtained from equation (40) and the shear flows can be found from equations (41) and (44). For panel row j = 0, the shear flow equations become

$$q_{i0} = \sum_{n=2}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} \frac{f_n(i+1) - f_n(i)}{2L \sin \frac{n\delta}{2}} \qquad (i \geqq 1)$$

and

$$q_{00} = \frac{3Q}{m} + \sum_{n=2}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} \left[\frac{f_n(1)}{L \sin \frac{n\delta}{2}} + \frac{2Q}{m\left(1 + \delta_{n,\frac{m}{2}}\right)} \right]$$

When the shear flows in panel row j = 0 are known, it is simpler to compute the remainder of the shear flows by use of the equations of statics rather than equations (41) and (44). In shear panels (0,1) and (0,-1) adjacent to the loaded panel, the shear flow is given by

$$q_{01} = q_{0,-1} = q_{00} - \frac{2p_{11} + QL}{L}$$

All the other shear flows are found by use of equation (10). If desired, the ring bending moments can be found from equations (42) and (45) and the ring thrust and transverse shear can be calculated from the formulas given in appendix B.

LIMITING CASE OF RIGID RINGS

If the ring bending stiffness is allowed to increase indefinitely, the rings approach complete rigidity in bending, the parameter C approaches zero, and a considerable simplification results. For this limiting case, equations (20) for the concentrated perturbation load reduce to

$$f_n(i+1) - 2\frac{B_n}{A_n} f_n(i) + f_n(i-1) = 0 \qquad (i \geq 1) \quad (47)$$

where

$$A_n = 3B\delta^2 - 1 + \cos n\delta$$

$$B_n = 3B\delta^2 + 2(1 - \cos n\delta)$$

This can be shown easily by multiplying equations (20) through by C and allowing C to approach 0. Equation (47) is a second-order finite-difference equation with constant coefficients. The same equation, together with its general solution, is given in reference 6, page 31. The solution compatible with the boundary conditions at infinity can be written as

$$f_n(i) = \alpha_n\left(\pm e^{-\lambda_n}\right)^i \qquad (48)$$

where

$$\cosh \lambda_n = \left|\frac{B_n}{A_n}\right|$$

and where the upper sign is taken when $A_n > 0$ and the lower sign when $A_n < 0$.

The arbitrary constant α_n is determined by evaluating the solution, equation (48), for $i = 0$ and introducing the value of $f_n(0)$ given in equation (9). The result is identical to α_{1n} given in equation (23)

$$\alpha_n = \frac{P}{m\left(1 + \delta_{n,\frac{m}{2}}\right)} \qquad (n \geq 2)$$

Equations (8) and (12), the expressions for stringer loads and shear flows, respectively, used before in the case of the concentrated perturbation load are still valid. The substitution into these expressions of the solution (48) with the constant α_n as found above yields the stringer loads and shear flows due to a concentrated perturbation load when the rings can be considered rigid.

For the case of the distributed perturbation load, equations (37) reduce in the limit to

$$(-A_n)f_n(2) + (2B_n + A_n)f_n(1) = 3B\delta^2 Sd_n$$

$$f_n(i+1) - 2\frac{B_n}{A_n} f_n(i) + f_n(i-1) = 0 \qquad (i \geqq 2)$$

The arbitrary constant α_n in the solution (48) is

$$\alpha_n = \frac{6B\delta^2}{A_n\left(\pm e^{-\lambda_n} + 1\right)} \frac{S}{m\left(1 + \delta_n, \frac{m}{2}\right)}$$

For the shear perturbation load, equations (46) reduce to

$$(-A_n)f_n(2) + (2B_n + A_n)f_n(1) = -6LQd_nB\delta^2 \sin\frac{n\delta}{2}$$

$$f_n(i+1) - 2\frac{B_n}{A_n} f_n(i) + f_n(i-1) = 0 \qquad (i \geqq 2)$$

The solution is again equation (48) and α_n becomes

$$\alpha_n = -\frac{12B\delta^2 \sin\frac{n\delta}{2}}{A_n\left(\pm e^{-\lambda_n} + 1\right)} \frac{QL}{m\left(1 + \delta_n, \frac{m}{2}\right)}$$

CONCLUDING REMARKS

Solutions have been obtained for the stress distributions caused by three types of loading on infinitely long, circular, semimonocoque cylinders with flexible rings. The solutions are adaptable to the construction of tables of influence coefficients which give stringer loads and shear flows in the neighborhood of the perturbation load due to a unit magnitude of that load. The perturbation loads are the tools of a method described in NACA Technical Note 3200 for the stress analysis of circular semimonocoque cylinders with cutouts.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., April 22, 1954.

APPENDIX A

SUMMARY OF SIGNIFICANT EQUATIONS

The formulas and parameters required for computing the stress distribution due to concentrated, distributed, and shear perturbation loads are collected in this appendix for reference.

Stringer Loads

Concentrated perturbation load (see fig. 1(a)):

$$p_{ij} = \frac{P}{2m} + \frac{P}{m}\cos j\delta + \sum_{n=2}^{\frac{m}{2}\text{ or }\frac{m-1}{2}} f_n(i)\cos nj\delta \qquad (i \geq 0)$$

where P is the applied load.

Distributed perturbation load (see fig. 1(b)):

$$p_{ij} = \frac{S}{2m} + \frac{S}{m}\cos j\delta + \sum_{n=2}^{\frac{m}{2}\text{ or }\frac{m-1}{2}} f_n(i)\cos nj\delta \qquad (i \geq 1)$$

where S is the total applied load.

Shear perturbation load (see fig. 1(c)):

$$p_{ij} = \sum_{n=2}^{\frac{m}{2}\text{ or }\frac{m-1}{2}} f_n(i)\sin n\left(j - \frac{1}{2}\right)\delta \qquad (i \geq 1)$$

Shear Flows

Concentrated perturbation load (see fig. 1(a)):

For the shear panels in panel row $j = 0$,

$$q_{i0} = \frac{p_{i0} - p_{i+1,0}}{2L}$$

and, for the remainder of the shear panels,

$$q_{ij} = \frac{p_{ij} - p_{i+1,j}}{L} + q_{i,j-1} \qquad (j \geqq 1)$$

Distributed perturbation load (see fig. 1(b)):

For the shear panel (0,0),

$$q_{00} = \frac{S - 2p_{10}}{2L}$$

for the remainder of the shear panels in panel row $j = 0$,

$$q_{i0} = \frac{p_{i0} - p_{i+1,0}}{2L} \qquad (i \geqq 1)$$

and, for all other shear panels,

$$q_{ij} = \frac{p_{ij} - p_{i+1,j}}{L} + q_{i,j-1} \qquad (j \geqq 1)$$

Shear perturbation load (see fig. 1(c)):

For the panel about which the load is applied,

$$q_{00} = \frac{3Q}{m} + \sum_{n=2}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} \left[\frac{f_n(1)}{L \sin \frac{n\delta}{2}} + \frac{2Q}{m\left(1 + \delta_{n,\frac{m}{2}}\right)} \right]$$

for the remainder of the shear panels in row $j = 0$,

$$q_{i0} = \sum_{n=2}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} \frac{f_n(i+1) - f_n(i)}{2L \sin \frac{n\delta}{2}} \qquad (i \geqq 1)$$

for the shear panel (0,1),

$$q_{01} = q_{00} - \frac{2p_{11} + QL}{L}$$

for the remainder of the shear panels in panel row j = 1,

$$q_{i1} = \frac{p_{i1} - p_{i+1,1}}{L} + q_{i0} \qquad (i \geqq 1)$$

and, for all other shear panels,

$$q_{ij} = \frac{p_{ij} - p_{i+1,j}}{L} + q_{i,j-1} \qquad (j \geqq 2)$$

Evaluation of the Trigonometric Coefficients $f_{\bar{n}}(i)$ for Flexible Rings

Basic parameters:

$$B = \frac{E}{G}\frac{A}{bt}\frac{R^2}{L^2}$$

$$C = \frac{AR^6}{IL^3b}$$

Auxiliary parameters:

$$\beta_n = 3 + \frac{4 + 3\dfrac{B\delta^2}{\sin^2\dfrac{n\delta}{2}}}{12CS_n}$$

$$\gamma_n = -2 + \frac{1 - \dfrac{3}{2}\dfrac{B\delta^2}{\sin^2\dfrac{n\delta}{2}}}{12CS_n}$$

Discriminating parameter:

$$D_n = \frac{2(\beta_n - 1)}{\gamma_n^2}$$

Trigonometric coefficients:

$$f_n(i) = \zeta_n^i \left[\alpha_{1n}\Lambda_{1n}(i) + \alpha_{2n}\Lambda_{2n}(i)\right] \qquad (n \geq 2)$$

where

$$\zeta_n = -\frac{\gamma_n}{|\gamma_n|} e^{-\psi_n}$$

$$\Lambda_{1n}(i) = \cos i\chi_n \qquad (D_n > 1)$$

$$= 1 \qquad (D_n = 1)$$

$$= \cosh i\chi_n \qquad (D_n < 1)$$

$$\Lambda_{2n}(i) = \sin i\chi_n \qquad (D_n > 1)$$

$$= i \qquad (D_n = 1)$$

$$= \sinh i\chi_n \qquad (D_n < 1)$$

$$\chi_n = \frac{1}{2}\cos^{-1}\left[\frac{\beta_n - 1}{2} - \sqrt{\left(\frac{\beta_n + 1}{2}\right)^2 - \gamma_n^2}\right] \qquad (D_n > 1)$$

$$= \frac{1}{2}\cosh^{-1}\left[\frac{\beta_n - 1}{2} - \sqrt{\left(\frac{\beta_n + 1}{2}\right)^2 - \gamma_n^2}\right] \qquad (D_n < 1)$$

$$\psi_n = \frac{1}{2}\cosh^{-1}\left[\frac{\beta_n - 1}{2} + \sqrt{\left(\frac{\beta_n + 1}{2}\right)^2 - \gamma_n^2}\right]$$

Arbitrary constants for concentrated perturbation load:

$$\alpha_{1n} = \frac{P}{m\left(1 + \delta_{n,\frac{m}{2}}\right)}$$

$$\alpha_{2n} = -\frac{\Theta_{1n} + 2(\gamma_n + 1)}{\Theta_{2n}} \frac{P}{m\left(1 + \delta_{n,\frac{m}{2}}\right)}$$

where P is the applied load and

$$\Theta_{sn} = \zeta_n^3 \Lambda_{sn}(3) + 2\gamma_n \zeta_n^2 \Lambda_{sn}(2) + (2\beta_n - 1)\zeta_n \Lambda_{sn}(1) \qquad (s = 1,2)$$

Arbitrary constants for distributed perturbation load:

$$\alpha_{1n} = \frac{\Gamma_{2n} \frac{\beta_n - 4\gamma_n - 2}{3} + \Omega_{2n}}{\Omega_{1n}\Gamma_{2n} - \Gamma_{1n}\Omega_{2n}} \frac{2S}{m\left(1 + \delta_{n,\frac{m}{2}}\right)}$$

$$\alpha_{2n} = -\frac{\Omega_{1n} + \Gamma_{1n} \frac{\beta_n - 4\gamma_n - 2}{3}}{\Omega_{1n}\Gamma_{2n} - \Gamma_{1n}\Omega_{2n}} \frac{2S}{m\left(1 + \delta_{n,\frac{m}{2}}\right)}$$

where S is the total applied load and

$$\Omega_{sn} = \zeta_n^3 \Lambda_{sn}(3) + (2\gamma_n - 1)\zeta_n^2 \Lambda_{sn}(2) + 2(\beta_n - \gamma_n)\zeta_n \Lambda_{sn}(1) \qquad (s = 1,2)$$

$$\Gamma_{sn} = \zeta_n^4 \Lambda_{sn}(4) + 2\gamma_n \zeta_n^3 \Lambda_{sn}(3) + 2\beta_n \zeta_n^2 \Lambda_{sn}(2) + (2\gamma_n - 1)\zeta_n \Lambda_{sn}(1) \qquad (s = 1,2)$$

Arbitrary constants for shear perturbation load:

$$\alpha_{1n} = -\frac{\Gamma_{2n}\frac{\beta_n - 4\gamma_n - 11}{3}\sin\frac{n\delta}{2}}{\Omega_{1n}\Gamma_{2n} - \Gamma_{1n}\Omega_{2n}}\frac{4QL}{m\left(1 + \delta_{n,\frac{m}{2}}\right)}$$

$$\alpha_{2n} = \frac{\Gamma_{1n}\frac{\beta_n - 4\gamma_n - 11}{3}\sin\frac{n\delta}{2}}{\Omega_{1n}\Gamma_{2n} - \Gamma_{1n}\Omega_{2n}}\frac{4QL}{m\left(1 + \delta_{n,\frac{m}{2}}\right)}$$

where Q is the applied load per unit length.

Evaluation of the Trigonometric Coefficients $f_n(i)$ for Rigid Rings

Basic parameter:

$$B = \frac{E}{G}\frac{A}{bt}\frac{R^2}{L^2}$$

Auxiliary parameters:

$$A_n = 3B\delta^2 - 1 + \cos n\delta$$

$$B_n = 3B\delta^2 + 2(1 - \cos n\delta)$$

$$\lambda_n = \cosh^{-1}\left|\frac{B_n}{A_n}\right|$$

Trigonometric coefficients:

$$f_n(i) = \alpha_n\left(\frac{A_n}{|A_n|}e^{-\lambda_n}\right)^i$$

Arbitrary constant for concentrated perturbation load:

$$\alpha_n = \frac{P}{m\left(1 + \delta_{n,\frac{m}{2}}\right)}$$

Arbitrary constant for distributed perturbation load:

$$\alpha_n = \frac{6B\delta^2}{A_n\left(\frac{A_n}{|A_n|}\, e^{-\lambda_n} + 1\right)} \frac{S}{m\left(1 + \delta_{n,\frac{m}{2}}\right)}$$

Arbitrary constant for shear perturbation load:

$$\alpha_n = -\frac{12B\delta^2 \sin\frac{n\delta}{2}}{A_n\left(\frac{A_n}{|A_n|}\, e^{-\lambda_n} + 1\right)} \frac{QL}{m\left(1 + \delta_{n,\frac{m}{2}}\right)}$$

APPENDIX B

BENDING MOMENT, AXIAL THRUST, AND TRANSVERSE SHEAR IN RINGS

Expressions will be developed for the bending moment, axial thrust, and transverse shear in a circular ring under tangential loads such as those which arise from the differences in shear flow across a ring in a circular semimonocoque cylinder.

Two cases must be considered: One case occurs with the concentrated and distributed perturbation loads, where the ring loading is antisymmetric about stringer $j = 0$. The other case occurs with the shear perturbation load, where the ring loading is symmetric about panel row $j = 0$.

Concentrated and distributed perturbation loads.- For the concentrated and distributed perturbation loads, the tangential loading on ring i has been written in the form of a finite trigonometric series (see eq. (13))

$$F_{ij} = q_{ij} - q_{i-1,j} = \sum_{n=2}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} b_{ni} \sin n\left(j + \frac{1}{2}\right)\delta \qquad \text{(B1)}$$

where

$$b_{ni} = -\frac{\Delta_{ii} f_n(i)}{2L \sin \frac{n\delta}{2}} \qquad (n \geq 2)$$

This ring load has a stepwise variation around the ring, being constant between stringers and having jump discontinuities at the stringers. The limitation that $n \geq 2$ ensures that the ring is in equilibrium.

The procedure will be to expand each term of the series (B1) in an infinite Fourier series in the variable ϕ. For each harmonic of the Fourier series, that is, for a continuous sinusoidal tangential force distribution on the ring, the moment, thrust, and shear in the ring are easily found (see ref. 5, p. 33, for example). On the basis of inextensional deformation and the neglect of transverse shear distortions, the results are as follows: If the tangential load on ring i is given by

$$\bar{a}_{ni} \cos n\phi + \bar{b}_{ni} \sin n\phi \qquad (n \geq 2)$$

then the moment, thrust, and shear in this ring are, respectively,

$$\left.\begin{aligned} M_n(i,\phi) &= -\bar{a}_{ni}\frac{R^2}{n(n^2-1)}\sin n\phi + \bar{b}_{ni}\frac{R^2}{n(n^2-1)}\cos n\phi \\ T_n(i,\phi) &= -\bar{a}_{ni}\frac{R}{n^2-1}\,n\sin n\phi + \bar{b}_{ni}\frac{R}{n^2-1}\,n\cos n\phi \\ V_n(i,\phi) &= \bar{a}_{ni}\frac{R}{n^2-1}\cos n\phi + \bar{b}_{ni}\frac{R}{n^2-1}\sin n\phi \end{aligned}\right\} \qquad (B2)$$

Figure 3 shows the sign convention used in writing equations (B2).

Consider, now, one term of the series (B1). To expand this term in a Fourier series, write

$$b_{ni}\sin n\left(j+\tfrac{1}{2}\right)\delta = \sum_{r=n}^{\infty}(c_r)_{ni}\sin r\phi \qquad (B3)$$

where the $(c_r)_{ni}$'s are the Fourier coefficients. It is obvious that the first harmonic which will occur in the Fourier series in equation (B3) must be that for which $r = n$. The other harmonics, then, will be added to this to build up the step shape of the loading function. The convention for measuring angle ϕ in this case is illustrated in figure 4(a). The index j can be thought of as a function of ϕ, that is: when $0 < \phi < \delta$, $j = 0$; when $\delta < \phi < 2\delta$, $j = 1$; and so forth.

In order to carry out the expansion, equation (B3) is multiplied through by $\sin l\phi$ and integrated from 0 to 2π

$$\sum_{j=0}^{m-1}\int_{j\delta}^{(j+1)\delta} b_{ni}\sin n\left(j+\tfrac{1}{2}\right)\delta\sin l\phi\,d\phi = \int_0^{2\pi}\sum_{r=n}^{\infty}(c_r)_{ni}\sin r\phi\sin l\phi\,d\phi$$

After integration, the right-hand side of this equation becomes

$$(c_l)_{ni}\pi$$

by virtue of the orthogonality of the trigonometric functions. The left-hand side becomes

$$\frac{2 \sin \frac{l\delta}{2}}{l} b_{ni} \sum_{j=0}^{m-1} \sin n\left(j + \frac{1}{2}\right)\delta \sin l\left(j + \frac{1}{2}\right)\delta$$

on carrying out the integration. From reference 4 it can be shown that

$$\sum_{j=0}^{m-1} \sin n\left(j + \frac{1}{2}\right)\delta \sin l\left(j + \frac{1}{2}\right)\delta = \frac{m}{2}\left[(-1)^{\frac{l-n}{m}} J_{\frac{l-n}{m}} - (-1)^{\frac{l+n}{m}} J_{\frac{l+n}{m}}\right]$$

where $J_r = 1$ if r is an integer, and $J_r = 0$ if r is not an integer.

Thus the Fourier coefficients are given by

$$(c_l)_{ni} = \frac{m}{\pi} b_{ni} \frac{\sin \frac{l\delta}{2}}{l}\left[(-1)^{\frac{l-n}{m}} J_{\frac{l-n}{m}} - (-1)^{\frac{l+n}{m}} J_{\frac{l+n}{m}}\right]$$

The nth term of the tangential loading on the ring is

$$b_{ni} \sin n\left(j + \frac{1}{2}\right)\delta = \frac{m}{\pi} b_{ni} \sum_{l=n}^{\infty}\left[(-1)^{\frac{l-n}{m}} J_{\frac{l-n}{m}} - (-1)^{\frac{l+n}{m}} J_{\frac{l+n}{m}}\right]\frac{\sin \frac{l\delta}{2}}{l} \sin l\phi \tag{B4}$$

By use of the properties of J_r this summation can be rewritten

$$b_{ni} \sin n\left(j + \frac{1}{2}\right)\delta = \frac{m}{\pi} b_{ni}\left[\sum_{r=0}^{\infty} (-1)^r \frac{\sin(rm + n)\frac{\delta}{2}}{rm + n} \sin(rm + n)\phi - \sum_{r=1}^{\infty} (-1)^r \frac{\sin(rm - n)\frac{\delta}{2}}{rm - n} \sin(rm - n)\phi\right] \tag{B5}$$

On expansion by the sum and difference formulas of trigonometry and with the use of the fact that $m\delta = 2\pi$, it is found that

$$\left.\begin{aligned} \sin(rm + n)\frac{\delta}{2} &= (-1)^r \sin\frac{n\delta}{2} \\ \sin(rm - n)\frac{\delta}{2} &= (-1)^{r+1} \sin\frac{n\delta}{2} \end{aligned}\right\} \tag{B6}$$

When equations (B6) are substituted into equation (B5), the following relationship results:

$$b_{ni} \sin n\left(j + \frac{1}{2}\right)\delta = \frac{m}{\pi} b_{ni} \sin\frac{n\delta}{2}\left[\sum_{r=0}^{\infty} \frac{\sin(rm + n)\phi}{rm + n} + \sum_{r=1}^{\infty} \frac{\sin(rm - n)\phi}{rm - n}\right]$$

$$= \frac{m}{\pi} b_{ni} \sin\frac{n\delta}{2} \sum_{r=-\infty}^{\infty} \frac{\sin(rm + n)\phi}{rm + n} \tag{B7}$$

From the first of equations (B2) it is seen that if the tangential loading on the ring is given by the right-hand side of equation (B7) then the bending moment in that ring is

$$M_n(i,\phi) = R^2 \frac{m}{\pi} b_{ni} \sin\frac{n\delta}{2} H_1(n,\phi) \tag{B8}$$

where

$$H_1(n,\phi) = \sum_{r=-\infty}^{\infty} \frac{\cos(rm + n)\phi}{(rm + n)^2\left[(rm + n)^2 - 1\right]}$$

Equation (B8) gives the bending moment in a ring which carries a tangential load distributed according to one term of the series of equation (B1). When the ring is loaded by the sum of such stepwise terms, as in equation (B1), then the moment is given by a sum of terms like (B8). So the bending moment in ring i is

$$M(i,\phi) = \sum_{n=2}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} R^2 \frac{m}{\pi} b_{ni} \sin\frac{n\delta}{2} H_1(n,\phi) \tag{B9}$$

For completeness, the expressions for axial thrust and transverse shear can be written in a similar manner -

$$T(i,\phi) = \sum_{n=2}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} R \frac{m}{\pi} b_{ni} \sin \frac{n\delta}{2} K_1(n,\phi)$$

$$V(i,\phi) = \sum_{n=2}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} R \frac{m}{\pi} b_{ni} \sin \frac{n\delta}{2} L_1(n,\phi)$$

where

$$K_1(n,\phi) = \sum_{r=-\infty}^{\infty} \frac{\cos(rm + n)\phi}{(rm + n)^2 - 1}$$

$$L_1(n,\phi) = \sum_{r=-\infty}^{\infty} \frac{\sin(rm + n)\phi}{(rm + n)\left[(rm + n)^2 - 1\right]}$$

Shear perturbation load.- In the case of the shear perturbation load, the tangential loading on ring i is given by the finite trigonometric series

$$F_{ij} = q_{ij} - q_{i-1,j} = \sum_{n=2}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} a_{ni} \cos nj\delta \qquad \text{(B10)}$$

where

$$a_{ni} = \frac{\Delta_{ii} f_n(i)}{2L \sin \frac{n\delta}{2}} \qquad (n \geq 2)$$

Equation (B10) can be treated in a manner analogous to the handling of equation (B1). That is, each term of the series in equation (B10) can be expanded in a Fourier series. Then the moment, thrust, and shear in the ring are written immediately.

Analogous to equation (B3), write—

$$a_{ni} \cos nj\delta = \sum_{r=n}^{\infty} (c_r)_{ni} \cos r\phi \qquad \text{(B11)}$$

where, now, the angle ϕ is as shown in figure 4(b). If both sides of equation (B11) are multiplied by cos $l\phi$ and integrated from 0 to 2π, there results for the Fourier coefficients:

$$(c_l)_{ni} = \frac{2 \sin \frac{l\delta}{2}}{\pi l} a_{ni} \sum_{j=0}^{m-1} \cos nj\delta \cos lj\delta$$

It can be shown (see ref. 4) that

$$\sum_{j=0}^{m-1} \cos nj\delta \cos lj\delta = \frac{m}{2}\left(J_{\frac{l-n}{m}} + J_{\frac{l+n}{m}}\right)$$

so the nth term of the tangential loading on the ring is

$$a_{ni} \cos nj\delta = \frac{m}{\pi} a_{ni} \sum_{l=n}^{\infty}\left(J_{\frac{l-n}{m}} + J_{\frac{l+n}{m}}\right)\frac{\sin \frac{l\delta}{2}}{l} \cos l\phi \qquad \text{(B12)}$$

This summation becomes

$$a_{ni} \cos nj\delta = \frac{m}{\pi} a_{ni} \sin \frac{n\delta}{2} \sum_{r=-\infty}^{\infty} (-1)^r \frac{\cos(rm + n)\phi}{rm + n}$$

which corresponds to equation (B7). Then the bending moment is

$$M(i,\phi) = - \sum_{n=2}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} R^2 \frac{m}{\pi} a_{ni} \sin \frac{n\delta}{2} H_2(n,\phi) \qquad \text{(B13)}$$

Similarly, thrust and shear are

$$T(i,\phi) = - \sum_{n=2}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} R \frac{m}{\pi} a_{ni} \sin \frac{n\delta}{2} K_2(n,\phi)$$

and

$$V(i,\phi) = \sum_{n=2}^{\frac{m}{2} \text{ or } \frac{m-1}{2}} R \frac{m}{\pi} a_{ni} \sin \frac{n\delta}{2} L_2(n,\phi)$$

where

$$H_2(n,\phi) = \sum_{r=-\infty}^{\infty} (-1)^r \frac{\sin(rm + n)\phi}{(rm + n)^2 \left[(rm + n)^2 - 1\right]}$$

$$K_2(n,\phi) = \sum_{r=-\infty}^{\infty} (-1)^r \frac{\sin(rm + n)\phi}{(rm + n)^2 - 1}$$

$$L_2(n,\phi) = \sum_{r=-\infty}^{\infty} (-1)^r \frac{\cos(rm + n)\phi}{(rm + n)\left[(rm + n)^2 - 1\right]}$$

APPENDIX C

EVALUATION OF DEFINITE INTEGRALS

In order to minimize the stress energy it is necessary to investigate the following definite integrals:

$$\int_0^{2\pi} H_1(n,\phi)H_1(l,\phi)d\phi = \int_0^{2\pi} \sum_{r=-\infty}^{\infty} D_{rn} \cos(rm + n)\phi \sum_{s=-\infty}^{\infty} D_{sl} \cos(sm + l)\phi \, d\phi \tag{C1}$$

and

$$\int_0^{2\pi} H_2(n,\phi)H_2(l,\phi)d\phi = \int_0^{2\pi} \sum_{r=-\infty}^{\infty} (-1)^r D_{rn} \sin(rm + n)\phi \sum_{s=-\infty}^{\infty} (-1)^s D_{sl} \sin(sm + l)\phi \, d\phi \tag{C2}$$

where

$$D_{rn} = \frac{1}{(rm + n)^2\left[(rm + n)^2 - 1\right]}$$

and where integers n and l are limited to the following ranges:

$$2 \leqq n \leqq \frac{m}{2}$$

$$2 \leqq l \leqq \frac{m}{2}$$

Consider the relation (C1). The right-hand side can be written

$$\frac{1}{2} \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} D_{rn}D_{sl}\left[\int_0^{2\pi} \cos(rm + n - sm - l)\phi \, d\phi + \int_0^{2\pi} \cos(rm + n + sm + l)\phi \, d\phi\right] = \frac{1}{2} \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} D_{rn}D_{sl}\left(\delta_{rm+n,sm+l} + \delta_{rm+n,-sm-l}\right)2\pi$$

Now, by virtue of the limited range of the integers n and l, the following relations can be written:

$$\delta_{rm+n,sm+l} = \delta_{s,r-\frac{l-n}{m}} = \delta_{sr}\delta_{ln}$$

$$\delta_{rm+n,-sm-l} = \delta_{s,-r-\frac{l+n}{m}} = \delta_{s,-r-1}\delta_{l,\frac{m}{2}}\delta_{n,\frac{m}{2}}$$

Thus, when $2 \leq n < \frac{m}{2}$, equation (C1) yields

$$\int_0^{2\pi} H_1(n,\phi)H_1(l,\phi)d\phi = 0 \qquad (l \neq n)$$

$$= \sum_{r=-\infty}^{\infty} D_{rn}^2\pi = S_n\pi \qquad (l = n)$$

If $n = \frac{m}{2}$, the following equation is obtained:

$$\int_0^{2\pi} H_1\left(\tfrac{m}{2},\phi\right)H_1\left(\tfrac{m}{2},\phi\right)d\phi = \sum_{r=-\infty}^{\infty}\left(D_{r,\frac{m}{2}}^2 + D_{r,\frac{m}{2}}D_{-r-1,\frac{m}{2}}\right)\pi$$

Since

$$D_{-r-1,\frac{m}{2}} = \frac{1}{\left(-rm - m + \frac{m}{2}\right)^2\left[\left(-rm - m + \frac{m}{2}\right)^2 - 1\right]}$$

$$= \frac{1}{\left(-rm - \frac{m}{2}\right)^2\left[\left(-rm - \frac{m}{2}\right)^2 - 1\right]}$$

$$= D_{r,\frac{m}{2}}$$

it is found that when $n = \frac{m}{2}$

$$\int_0^{2\pi} H_1\left(\frac{m}{2},\phi\right)H_1\left(\frac{m}{2},\phi\right)d\phi = 2\sum_{r=-\infty}^{\infty} D_{r,\frac{m}{2}}{}^2\pi = 2S_{\frac{m}{2}}\pi$$

To summarize, then,

$$\int_0^{2\pi} H_1(n,\phi)H_1(l,\phi)d\phi = 0 \qquad (l \neq n)$$

$$= \delta_{n}\pi\left(1 + \delta_{n,\frac{m}{2}}\right) \qquad (l = n)$$

Consider the relation (C2). It is handled in a manner analogous to the treatment of (C1). Equation (C2) can be written

$$\int_0^{2\pi} H_2(n,\phi)H_2(l,\phi)d\phi = \sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}(-1)^r D_{rn}(-1)^s D_{s\,l}\frac{1}{2}\left(\delta_{rm+n,sm+l} - \delta_{rm+n,-sm-l}\right)2\pi$$

$$= \sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}(-1)^r(-1)^s D_{rn}D_{s\,l}\left(\delta_{sr}\delta_{nl} - \delta_{s,-r-l}\delta_{n,\frac{m}{2}}\delta_{l,\frac{m}{2}}\right)\pi$$

For $2 \leq n < \frac{m}{2}$,

$$\int_0^{2\pi} H_2(n,\phi)H_2(l,\phi)d\phi = 0 \qquad (l \neq n)$$

$$= \sum_{r=-\infty}^{\infty} D_{rn}{}^2\pi = S_n\pi \qquad (l = n)$$

If $n = \frac{m}{2}$,

$$\int_0^{2\pi} H_2\left(\frac{m}{2},\phi\right)H_2\left(\frac{m}{2},\phi\right)d\phi = \sum_{r=-\infty}^{\infty}\left[D_{r,\frac{m}{2}}^{\ 2} - (-1)^r(-1)^{-r-1}D_{r,\frac{m}{2}}\,D_{-r-1,\frac{m}{2}}\right]\pi$$

$$= \sum_{r=-\infty}^{\infty}\left[D_{r,\frac{m}{2}}^{\ 2} - (-1)^{-1}D_{r,\frac{m}{2}}^{\ 2}\right]\pi$$

$$= 2\sum_{r=-\infty}^{\infty} D_{r,\frac{m}{2}}^{\ 2}\pi$$

$$= 2S_{\frac{m}{2}}\pi$$

Thus the definite integral (C2) gives precisely the same result as (C1)

$$\int_0^{2\pi} H_2(n,\phi)H_2(l,\phi)d\phi = 0 \qquad (l \neq n)$$

$$= S_n\pi\left(1 + \delta_{n,\frac{m}{2}}\right) \qquad (l = n)$$

The sum

$$S_n = \sum_{r=-\infty}^{\infty} D_{rn}^{\ 2} = \sum_{r=-\infty}^{\infty} \frac{1}{(rm + n)^4\left[(rm + n)^2 - 1\right]^2}$$

can be expressed in closed form with the aid of formula 6.495, number 2, reference 7. The result is

$$S_n = \frac{\delta^4}{12}\frac{2 + \cos n\delta}{(1 - \cos n\delta)^2} + \frac{\delta^2}{1 - \cos n\delta} - \frac{\delta^2}{4}\frac{\cos n\delta \cos \delta - 1}{(\cos n\delta - \cos \delta)^2} + \frac{5}{4}\frac{\delta \sin \delta}{\cos n\delta - \cos \delta}$$

However, the series form of S_n, because of its rapid convergence, may be more convenient than the closed form for use in computation.

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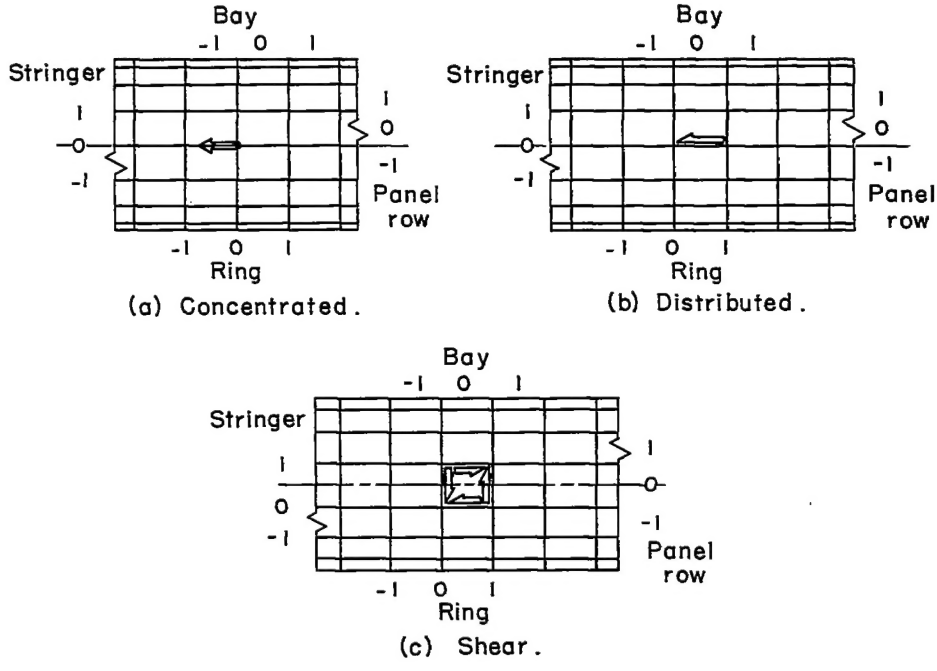


Figure 1.- Perturbation loads.

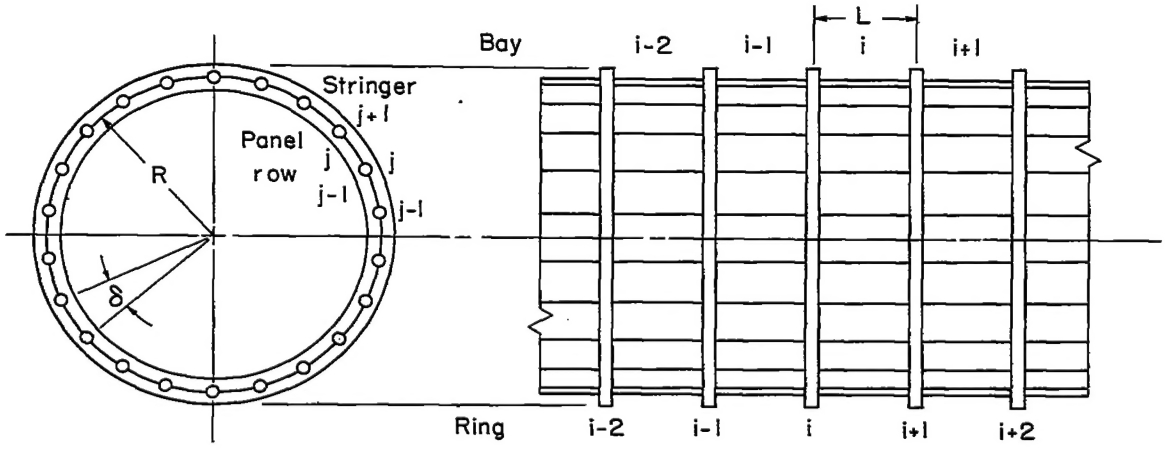
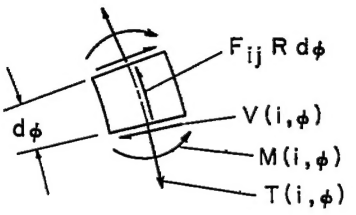


Figure 2.- Portion of typical cylinder.



Detail of ring element.

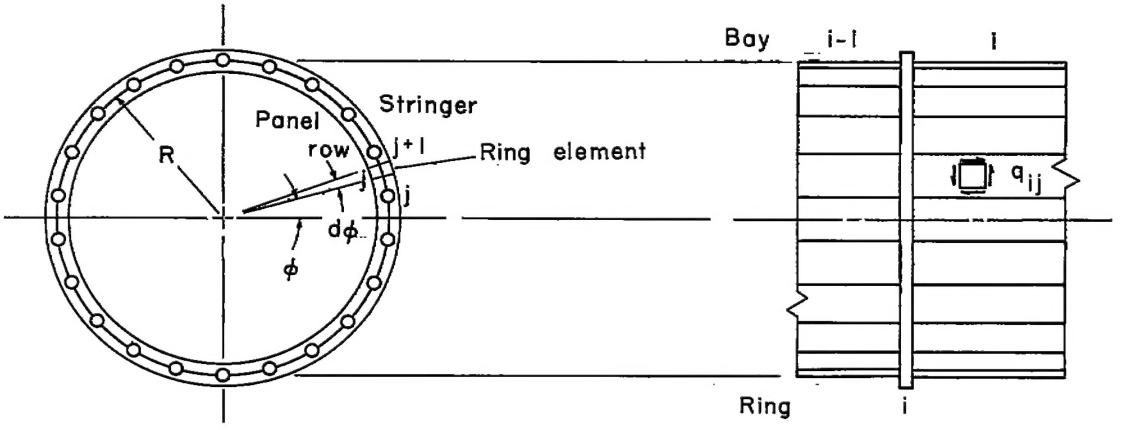
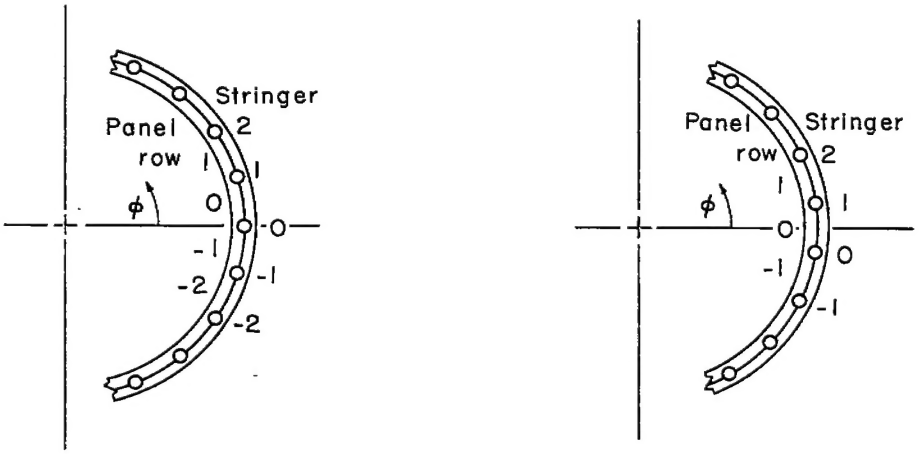


Figure 3.- Positive sense of quantities used in analysis.



(a) For concentrated and distributed perturbation loads.

(b) For shear perturbation load.

Figure 4.- Conventions for angular coordinate ϕ.

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